

# Asymptotic stability of strong contact discontinuity for full compressible Navier-Stokes equations with initial boundary value problem

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**Abstract.** This paper is concerned with Dirichlet problem  $u(0, t) = 0$ ,  $\theta(0, t) = \theta_-$  for one-dimensional full compressible Navier-Stokes equations in the half space  $\mathbb{R}_+ = (0, +\infty)$ . Because the boundary decay rate is hard to control, stability of contact discontinuity result is very difficult. In this paper, we raise the decay rate and establish that for a certain class of large perturbation, the asymptotic stability result is contact discontinuity. Also, we ask the strength of contact discontinuity not small. The proofs are given by the elementary energy method.

**Keywords:** Strong contact discontinuity, Dirichlet problem, Navier-Stokes equations, Asymptotic stability

## 1 Introduction

We consider one-dimensional compressible viscous heat-conducting flow in the half space  $\mathbb{R}_+ = [0, \infty)$ , which is governed by the following initial-boundary value problem in Eulerian coordinate  $(\tilde{x}, t)$ :

$$\left\{ \begin{array}{l} \tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = 0, \quad (\tilde{x}, t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ (\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p})_{\tilde{x}} = \mu\tilde{u}_{\tilde{x}\tilde{x}}, \\ \left( \tilde{\rho} \left( \tilde{e} + \frac{\tilde{u}^2}{2} \right) \right)_t + \left( \tilde{\rho}\tilde{u} \left( \tilde{e} + \frac{\tilde{u}^2}{2} \right) + \tilde{p}\tilde{u} \right)_{\tilde{x}} = \kappa\tilde{\theta}_{\tilde{x}\tilde{x}} + (\mu\tilde{u}\tilde{u}_{\tilde{x}})_{\tilde{x}}, \\ (\tilde{\rho}, \tilde{u}, \tilde{\theta})|_{\tilde{x}=0} = (\rho_-, 0, \theta_-), \\ (\tilde{\rho}, \tilde{u}, \tilde{\theta})|_{t=0} = (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)(\tilde{x}) \rightarrow (\rho_+, 0, \theta_+) \quad \text{as } \tilde{x} \rightarrow \infty, \end{array} \right. \quad (1.1)$$

where  $\tilde{\rho}$ ,  $\tilde{u}$  and  $\tilde{\theta}$  are the density, the velocity and the absolute temperature, respectively, while  $\mu > 0$  is the viscosity coefficient and  $\kappa > 0$  is the heat-conductivity coefficients, respectively. It is assumed throughout the paper that  $\rho_{\pm}$  and  $\theta_{\pm}$  are prescribed positive constants. We shall focus our interest on the polytropic ideal gas with  $|\theta_+ - \theta_-|$  and  $|\rho_+ - \rho_-|$  are general constants, so the pressure  $\tilde{p} = \tilde{p}(\tilde{\rho}, \tilde{\theta})$  and the internal energy  $\tilde{e} = \tilde{e}(\tilde{\rho}, \tilde{\theta})$  are related by the second law of thermodynamics:

$$\tilde{p} = R\tilde{\rho}\tilde{\theta}, \quad \tilde{e} = \frac{R}{\gamma - 1}\tilde{\theta} + const., \quad (1.2)$$

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where  $\gamma > 1$  is the adiabatic exponent and  $R > 0$  is the gas constant.

The boundary condition (1.1)<sub>4</sub> implies that, through the boundary  $\tilde{x} = 0$ , the fluid with density  $\rho_-$  flows into the region  $\mathbb{R}_+$  at the speed  $u = 0$ . So the initial-boundary value problem (1.1) is the so-called *impermeable wall* problem. In terms of various boundary values, Matsumura [9] classified all possible large-time behaviors of the solutions for the one-dimensional (isentropic) compressible Navier-Stokes equations.

To state our main results we first transfer (1.1) to the problem in the Lagrangian coordinate and then make use of a coordinate transformation to reduce the initial-boundary value problem (1.1) into the following form:

$$\begin{cases} v_t - u_x = 0, & (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ u_t + \left( \frac{R\theta}{v} \right)_x = \mu \left( \frac{u_x}{v} \right)_x, \\ \frac{R}{\gamma - 1} \theta_t + R \frac{\theta}{v} u_x = \kappa \left( \frac{\theta_x}{v} \right)_x + \mu \frac{u_x^2}{v}, \\ (v, u, \theta)|_{x=0} = (v_-, 0, \theta_-), \quad t > 0, \\ (v, u, \theta)|_{t=0} = (v_0, u_0, \theta_0) \rightarrow (v_+, 0, \theta_+) \quad \text{as } x \rightarrow \infty, \end{cases} \quad (1.3)$$

where  $v_{\pm}$  and  $\theta_{\pm}$  are given positive constants, and  $v_0, \theta_0 > 0$ . In fact  $v = 1/\rho(x, t)$ ,  $u = u(x, t)$ ,  $\theta = \theta(x, t)$  and  $R\theta/v = p(v, \theta)$  are the specific volume, velocity, temperature and pressure as in (1.1).

There have been a lot of works on the asymptotic behaviors of solutions to the initial-boundary value (or Cauchy) problem for the Navier-Stokes equations toward basic waves or their viscous versions, see, for example, [3–26] and the reference therein. In terms of various boundary values, Matsumura and Nishihara [8] classified all possible large-time behaviors of the solutions for one-dimensional (isentropic) compressible Navier-Stokes equations. In the case where  $u(0, t) = 0$ , the problem is called *impermeable wall* problem. Inflow problem is one of Dirichlet problems with  $u(0, t) < 0$ . Matsumura and Nishihara [8] have obtained the stability theorems on both the boundary layer solution and the superposition of a boundary-layer solution and a rarefaction wave for inflow problem. Due to Huang et al. [19] in which the asymptotic stability on both the viscous shock wave and the superposition of a viscous shock wave and a boundary-layer solution are studied. However, the problem of stability of contact discontinuities are associated with linear degenerate fields and previous results are less stable than the nonlinear waves for the inviscid system (Euler equations). It was observed in [13, 15], where the metastability of contact waves was studied for viscous conservation laws with artificial viscosity, that the contact discontinuity cannot be the asymptotic state for the viscous system, and a diffusive wave, which approximated the contact discontinuity on any finite time interval, actually dominates the large-time behavior of solutions. The nonlinear stability of weak contact discontinuity for the (full) compressible Navier-Stokes equations was then investigated in [20, 21] for the free boundary value problem, [22, 23] for the Cauchy problem and [28, 31] for the inflow problem. In [22] they point out because of the decay rate problem, viscous contact discontinuity for Dirichlet impermeable wall problem becomes difficulty.

Except Dirichlet impermeable wall problem, recently, some problems are call stability of strong viscous waves (see [18]–[27], [30]). These stability results (to Cauchy problem or free boundary problem) are shown with some special conditions. Especially, zero dissipation result is shown in [26] and  $\gamma \rightarrow 1$  in [18] or [25]. Base on small oscillation, initial smallness perturbation

or zero dissipation (and so on), Navier-Stokes equations stability results can be obtained with some skills.

The main purpose of this paper is to improve the previous studies for Dirichlet problem . Base on the new estimates on the heat kernel in [29], the decay rate can be promote. Then, we justify that the solution  $(v, u, \theta)$  of the Navier-Stokes system (1.3) tends asymptotically to contact discontinuity which is the Riemann solution (2.2) in  $\mathbb{R}_+$ . For the strength of the contact discontinuity is not small, we call it strong contact discontinuity . Also, we get rid off smallness  $\|(\varphi_{0x}, \psi_{0x}, \zeta_{0x})\|_{L^2}$  and  $\gamma \rightarrow 1$ , i.e., for the Direchlet problem (1.3), it is possible to be resolved and stable and the solution approximate the strong contact discontinuity .

**Notation.** Throughout this paper, we shall denote  $H^l(\mathbb{R}_+)$  the usual  $l$  –  $th$  order Sobolev space with the norm

$$\|f\|_l = \left( \sum_{j=0}^l \|\partial_x^j f\|^2 \right)^{1/2}, \quad \|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}_+)}.$$

For simplicity, we also use  $C$  or  $C_i$  ( $i = 1, 2, 3, \dots$ ) to denote the various positive generic constants.  $C(\delta_0)$  stands for suitably small constant about  $\delta_0$  and  $C_v = \frac{R}{\gamma - 1} \cdot \epsilon$  and  $\epsilon_i$  ( $i = 1, 2, 3, \dots$ ) stand for suitably small positive constant in Cauchy-Schwarz inequality and  $\partial_x^i = \frac{\partial^i}{\partial x^i}$ .

## 2 Reformulation and main result

As shown in previous studies, the asymptotic behavior is well characterized by the solutions to the corresponding Riemann problem for the hyperbolic part of (1.3) (that is, Euler system):

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v, \theta)_x = 0, \\ \frac{R}{\gamma - 1} \theta_t + R \frac{\theta}{v} u_x = 0, \\ (v, u, \theta)(x, 0) = (v_-, 0, \theta_-) \quad \text{if } x < 0, \\ (v, u, \theta)(x, 0) = (v_+, 0, \theta_+) \quad \text{if } x > 0, \end{cases} \quad (2.1)$$

The Riemann problem of system (2.1) admits a contact discontinuity

$$(\overline{V}, \overline{U}, \overline{\Theta}) = \begin{cases} (v_-, 0, \theta_-), & x < 0, \\ (v_+, 0, \theta_+), & x > 0, \end{cases} \quad (2.2)$$

provided that

$$p_- = R \frac{\theta_-}{v_-} = p_+ = R \frac{\theta_+}{v_+}. \quad (2.3)$$

As that in [21] we conjecture a pair of  $(V, U, \Theta)(x, t)$  is as follows

$$P(V, \Theta) = R \frac{\Theta}{V} = p_+, \quad U(x, t) = \frac{\kappa(\gamma - 1)\Theta_x}{\gamma R \Theta}, \quad (2.4)$$

and

$$\begin{cases} \Theta_t = a(\ln \Theta)_{xx}, & a = \frac{\kappa p_+(\gamma - 1)}{\gamma R^2} > 0, \\ \Theta(0, t) = \theta_-, \\ \Theta(x, 0) = \Theta_0(x) \rightarrow \theta_+, & \text{as } x \rightarrow +\infty, \end{cases} \quad (2.5)$$

with  $\Theta_0(x) = \left( \frac{2}{\sqrt{\pi}}(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}) \int_0^{\ln(x+\sqrt{1+x^2})} \exp\{-y^2\} dy + \theta_-^{1/\delta_0} \right)^{\delta_0}$ . Here  $\delta_0$  is a suitably small constant and  $1/\delta_0$  is an integer.

According to the smallness  $\delta_0$ , we can find  $\Theta_0(x)$  is nearly like a function  $f(x) = \begin{cases} \theta_+, & x > 0; \\ \theta_-, & x = 0. \end{cases}$  That means  $\Theta_0(x)$  satisfying the following properties.

**Lemma 2.1**

$$\begin{aligned} \|\Theta_{0x}\|_{L^1} &\leq C, \quad |\Theta_{0x}| \leq C\delta_0, \quad |\Theta_{0xx}| \leq C\delta_0, \quad \|\Theta_{0x}\|^2 \leq C\delta_0^2, \\ \|\Theta_{0xx}\|^2 &\leq C\delta_0^2, \quad \|\Theta_{0xxx}\|^2 \leq C, \quad \|\Theta_0 - \theta_+\|_{L^1} \leq C. \end{aligned} \quad (2.6)$$

*Proof.* In fact, if  $K(x) = \ln(x + \sqrt{1+x^2})$ , we can get

$$\begin{aligned} \int_{\mathbb{R}_+} \exp\{-K^2(x)\} dx &= \int_{\mathbb{R}_+} \exp\{-K^2(x)\} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} dx \\ &= \int_{\mathbb{R}_+} \exp\{-K^2(x)\} \sqrt{1+x^2} dK(x) \\ &\leq \int_{\mathbb{R}_+} \exp\{-K^2(x)\} \exp\{K(x)\} dK(x) \\ &\leq \int_{\mathbb{R}_+} \exp\{-K^2(x) + K(x)\} dK(x) \leq C. \end{aligned} \quad (2.7)$$

Set

$$H(x) = \Theta_0^{1/\delta_0} = \frac{\theta_+^{1/\delta_0} + \theta_-^{1/\delta_0}}{2} + \frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{\sqrt{\pi}} \int_0^{K(x)} \exp\{-x^2\} dx, \quad (2.8)$$

from  $K_x = (1+x^2)^{-1/2}$  and

$$\Theta_{0x} = \delta_0 H^{\delta_0-1}(x) H_x(x) = \delta_0 H(x)^{\delta_0-1} \frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{\sqrt{\pi}} K_x(x) \exp\{-K^2(x)\} > 0, \quad x \in \mathbb{R}_+, \quad (2.9)$$

we can get  $\|\Theta_{0x}\|_{L^1(\mathbb{R}_+)} < C$ .

When  $x > 0$ ,  $K(x) > 0$ , we can know

$$\frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{H(x)} \leq C \frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{\theta_+^{1/\delta_0} + \theta_-^{1/\delta_0}} \leq C. \quad (2.10)$$

Because

$$\theta_-^{1/\delta_0} \leq H(x) \leq \theta_+^{1/\delta_0}, \quad (2.11)$$

from (2.9) and  $K_x(x) = \frac{1}{\sqrt{1+x^2}}$ , we can get  $|\Theta_{0x}| \leq C\delta_0$ . Also from (3.1), (3.2), (2.9) and (3.3) we can get  $\|\Theta_{0x}\| \leq C\delta_0$ . Similar as above estimates, it is easy to check  $\|\Theta_{0xx}\| \leq C\delta_0^2$  and  $\|\Theta_{0xxx}\| \leq C$ .

When  $\delta_0 = \frac{1}{2k+1}$ ,  $k \in \{1, 2, 3, \dots\}$  is a suitably large constant, from the equality  $a^n - b^n = (a-b) \sum_{i=0}^{n-1} a^{n-1-i} b^i$ ,  $\forall a > 0, b > 0, n \in \{1, 2, 3, \dots\}$  and (3.1), (2.9), (3.3), we can get that

$$\begin{aligned}
\int_{\mathbb{R}_+} |\Theta_0 - \theta_+| dx &= \int_{\mathbb{R}_+} |H^{\delta_0} - \theta_+| dx \\
&= \int_{\mathbb{R}_+} \frac{|H - \theta_+^{1/\delta_0}|}{\sum_{i=0}^{2k} H^{(2k-i)/(2k+1)} \theta_+^{i/(2k+1)}} dx \\
&\leq C \int_{\mathbb{R}_+} \frac{(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}) \exp\{-CK^2(x)\}}{\sum_{i=0}^{2k} H^{(2k-i)/(2k+1)} \theta_+^{i/(2k+1)}} dx \\
&\leq C(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}) \sup_{x \in \mathbb{R}_+} H^{1/(2k+1)-1} \leq C(\theta_+ + \theta_-).
\end{aligned} \tag{2.12}$$

□

In summary we have constructed a pair of functions  $(V, U, \Theta)$  such that

$$\left\{ \begin{aligned}
R \frac{\Theta}{V} &= p_+, \\
V_t &= U_x, \\
U_t + P(V, \Theta)_x &= \mu \left( \frac{U_x}{V} \right)_x + F, \\
\frac{R}{\gamma-1} \Theta_t + R \frac{\Theta}{V} U_x &= \kappa \left( \frac{\Theta_x}{V} \right)_x + \mu \frac{U_x^2}{V} + G, \\
(V, U, \Theta)(0, t) &= (v_-, \frac{\kappa(\gamma-1)}{\gamma R} \frac{\Theta_x}{\Theta} |_{x=0}, \theta_-), \\
(V, U, \Theta)(x, 0) &= (V_0, U_0, \Theta_0) = (\frac{R}{p_+} \Theta_0, \frac{\kappa(\gamma-1)}{\gamma R} \frac{\Theta_{0x}}{\Theta_0}, \Theta_0) \rightarrow (v_+, 0, \theta_+), \text{ as } x \rightarrow +\infty,
\end{aligned} \right. \tag{2.13}$$

where

$$\begin{aligned}
G(x, t) &= -\mu \frac{U_x^2}{V} = O((\ln \Theta)_{xx}^2), \\
F(x, t) &= \frac{\kappa(\gamma-1)}{\gamma R} \left\{ (\ln \Theta)_{xt} - \mu \left( \frac{(\ln \Theta)_{xx}}{V} \right)_x \right\} \\
&= \frac{\kappa a(\gamma-1) - \mu p_+ \gamma}{R\gamma} \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x.
\end{aligned} \tag{2.14}$$

Denote

$$\varphi(x, t) = v(x, t) - V(x, t),$$

$$\begin{aligned}\psi(x, t) &= u(x, t) - U(x, t), \\ \zeta(x, t) &= \theta(x, t) - \Theta(x, t).\end{aligned}\tag{2.15}$$

Combining (2.13) and (1.3), the original problem can be reformulated as

$$\begin{cases} \varphi_t = \psi_x, \\ \psi_t - (\frac{R\Theta}{vV}\varphi)_x + (\frac{R\zeta}{v})_x = -\mu(\frac{U_x}{vV}\varphi)_x + \mu(\frac{\psi_x}{v})_x - F, \\ \frac{R}{\gamma-1}\zeta_t + \frac{R\theta}{v}(\psi_x + U_x) - \frac{R\Theta}{V}U_x = \kappa(\frac{\zeta_x}{v})_x - \kappa(\frac{\Theta_x\varphi}{vV})_x + \mu(\frac{u_x^2}{v} - \frac{U_x^2}{V}) - G, \\ (\varphi, \psi, \zeta)(0, t) = (0, u_b - U(0, t), 0) = (0, -\kappa(\gamma-1)\Theta_x(0, t)/(\gamma R\theta_-), 0), \\ (\varphi, \psi, \zeta)(x, 0) = (\varphi_0, \psi_0, \zeta_0) = (v_0 - V_0, u_0 - U_0, \theta_0 - \Theta_0), \end{cases}\tag{2.16}$$

Under the above preparation in hand, we assume throughout of this section that

$$(\varphi_0, \zeta_0)(x) \in H_0^1(0, \infty), \quad \psi_0(x) \in H^1(0, \infty).$$

Moreover, for an interval  $I \in [0, \infty)$ , we define the function space

$$X(I) = \{(\varphi, \psi, \zeta) \in C(I, H^1) | \varphi_x \in L^2(I; L^2), (\psi_x, \zeta_x) \in L^2(I; H^1)\}.$$

Our main results of this paper now reads as follows.

**Theorem 2.1** *If  $(v_0 - v_+, u_0, \theta_0 - \theta_+) \in H^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ ,  $\|(v_0 - \bar{V}, u_0 - \bar{U}, \theta_0 - \bar{\Theta})\|$  suitably small,  $\frac{v_-}{\theta_-} = \frac{v_+}{\theta_+}$  and  $|\theta_+ - \theta_-|$  not small, (1.3) has a global solution  $(v, u, \theta)$  satisfying  $(\varphi, \psi, \zeta) \in X([0, \infty))$ , and when  $t \rightarrow \infty$ ,*

$$\|(v - \bar{V}, u - \bar{U}, \theta - \bar{\Theta})\|_{L^\infty(\mathbb{R}_+)} \rightarrow (0, 0, 0).$$

### 3 Preliminary

In this section, to study the asymptotic behavior of the solution to the Cauchy problem (1.3), we provide some preliminary lemmas and list the a priori estimate that are important for the proof of Theorem 2.1.

**Lemma 3.1** *If  $\delta_0$  and  $\Theta_0$  satisfying the condition in Theorem 2.1 and*

$$\theta_2(x, t) = \int_0^{+\infty} (4\pi at)^{-1/2} (\Theta_0(h) - \theta_-) \left\{ \exp\left\{-\frac{(h-x)^2}{4at}\right\} - \exp\left\{-\frac{(h+x)^2}{4at}\right\} \right\} dh + \theta_-,$$

*we can get*

$$\begin{aligned}\theta_{2t} &= a\theta_{2xx}; \\ \theta_2(0, t) &= \theta_-; \\ \theta_2(x, 0) &= \theta_{20}(x) = \begin{cases} \Theta_0(x) \rightarrow \theta_+, & x > 0; \\ -\Theta_0(-x) + 2\theta_- \rightarrow 2\theta_- - \theta_+, & x \leq 0, \end{cases}\end{aligned}\tag{3.1}$$

*and*

$$\int_0^t \|\theta_{2x}\|^2 dt \leq C(1+t)^{1/3},\tag{3.2}$$

*Proof.* Because  $\theta_2(x, t)$  can be rewrite to

$$\theta_2(x, t) = \int_{-\infty}^{+\infty} (4\pi at)^{-1/2} \theta_{20}(h) \exp\left\{-\frac{(x-h)^2}{4at}\right\} dh,$$

and  $\theta_{20}(x) \in C^1(\mathbb{R})$ , we find that  $\theta_2(x, t)$  is a fundamental solution of (3.1), it is easy to check  $\lim_{t \rightarrow 0} \theta_2(x, t) = \theta_{20}(x)$ , so we finish (3.1).

Because

$$\begin{aligned} \theta_{2x} &= \int_0^{+\infty} (4\pi at)^{-1/2} (\Theta_0(z) - \theta_-) \left\{ \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{z-x}{2at} + \exp\left\{-\frac{(z+x)^2}{4at}\right\} \frac{z+x}{2at} \right\} dz \\ &= \int_0^{+\infty} (4\pi at)^{-1/2} \Theta_{0z}(z) \left\{ \exp\left\{-\frac{(z-x)^2}{4at}\right\} - \exp\left\{-\frac{(z+x)^2}{4at}\right\} \right\} dz \\ &= \int_0^{+\infty} (4\pi at)^{-1/2} (\Theta_0(z) - \theta_+ + \theta_+ - \Theta_0(x)) \\ &\quad \times \left\{ \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{z-x}{2at} + \exp\left\{-\frac{(z+x)^2}{4at}\right\} \frac{z+x}{2at} \right\} dz, \end{aligned} \quad (3.3)$$

By using Hölder inequality, Fubini Theorem and  $\|\Theta_0 - \theta_+\|_{L^1(\mathbb{R}_+)} < C$ , we can get from (3.3) that

$$\begin{aligned} \int_0^t \int_0^\infty \theta_{2x}^2 dx dt &\leq C \int_0^t \int_0^\infty (4\pi at)^{-1} \left\{ \int_0^\infty \Theta_{0z} \left( \exp\left\{-\frac{(z-x)^2}{4at}\right\} - \exp\left\{-\frac{(z+x)^2}{4at}\right\} \right) dz \right\}^2 dx dt \\ &\leq C \int_0^t \int_0^\infty (4\pi at)^{-1} \int_0^\infty |\Theta_{0z}| \left\{ \exp\left\{-\frac{(z-x)^2}{4at}\right\} + \exp\left\{-\frac{(z+x)^2}{4at}\right\} \right\} dz \\ &\quad \times \int_0^{+\infty} (4\pi at)^{-1/2} |\Theta_0(z) - \theta_+ + \theta_+ - \Theta_0(x)| \left\{ \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{|z-x|}{2at} \right. \\ &\quad \left. + \exp\left\{-\frac{(z+x)^2}{4at}\right\} \frac{|z+x|}{2at} \right\} dz dx dt \\ &\quad + C \int_0^1 \int_0^\infty (4\pi at)^{-1/2} \int_0^\infty |\Theta_{0z}| \left\{ \exp\left\{-\frac{(z-x)^2}{4at}\right\} + \exp\left\{-\frac{(z+x)^2}{4at}\right\} \right\} dz dx dt \\ &\leq C \int_1^t \int_0^\infty (4\pi at)^{-1} \int_0^{+\infty} (4\pi at)^{-1/2} |\Theta_0(z) - \theta_+| \\ &\quad \times \left\{ \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{|z-x|}{2at} + \exp\left\{-\frac{(z+x)^2}{4at}\right\} \frac{|z+x|}{2at} \right\} dx dz dt \\ &\quad + C \int_1^t \int_0^\infty (4\pi at)^{-1} \int_0^{+\infty} (4\pi at)^{-1/2} |\theta_+ - \Theta_0(x)| \\ &\quad \times \left\{ \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{|z-x|}{2at} + \exp\left\{-\frac{(z+x)^2}{4at}\right\} \frac{|z+x|}{2at} \right\} dz dx dt + C \\ &\leq C \|\Theta_0 - \theta_+\|_{L^1(\mathbb{R}_+)} \ln(1+t) + C \leq C(1+t)^{1/3}. \end{aligned}$$

So we finish this lemma.  $\square$

From (2.4) and (2.13), we obtain

$$(|V_x| + |U|) \leq C|\Theta_x|, \quad |\Theta_x|^2 \leq C\|(\ln \Theta)_x\| \|(\ln \Theta)_{xx}\|, \quad |U_x|^2 \leq C\|(\ln \Theta)_{xx}\| \|(\ln \Theta)_{xxx}\|. \quad (3.4)$$

According to the definition of  $(\overline{V}, \overline{U}, \overline{\Theta})$  in (2.2), when the time  $t \rightarrow \infty$  and  $x \in \mathbb{R}_+$ , it is easily check that  $(V, U, \Theta)$  is nearly close to the contact discontinuity  $(\overline{V}, \overline{U}, \overline{\Theta})$  by the following lemma.

**Lemma 3.2** *There exist a positive constant  $C$  such that*

$$\|(\ln \Theta)_x\|^2 + a \int_0^t \|(\ln \Theta)_{xx}\|^2 dt \leq C\delta_0^2. \quad (3.5)$$

(see(3.13)–(3.14))

$$\|\Theta - \theta_2\|^2 + \int_0^t \|(\ln \Theta)_x\|^2 dt \leq C(1+t)^{1/3}. \quad (3.6)$$

(see(3.15)–(3.16))

$$\|(\ln \Theta)_x\|^2 \leq C(1+t)^{-2/3}. \quad (3.7)$$

(see(3.17)–(3.20))

$$\|(\ln \Theta)_{xx}\|^2 \leq C(1+t)^{-5/3}. \quad (3.8)$$

(see(3.21)–(3.25))

$$\|(\ln \Theta)_{xx}\|^2(1+t) + \int_0^t \|\partial_x^3 \ln \Theta\|^2(1+t) dt \leq C\delta_0^2. \quad (3.9)$$

(see(3.26))

$$\|\partial_x^3 \ln \Theta\|^2 \leq C(1+t)^{-8/3}. \quad (3.10)$$

(see(3.27)–(3.29))

$$\|\Theta - \theta_+\|_{L^\infty}^2 \leq C\delta_0^{1/4}(1+t)^{-1/24}. \quad (3.11)$$

(see(3.31)–(3.33))

*Proof.* From (2.5) we know

$$(\ln \Theta)_t = a \frac{(\ln \Theta)_{xx}}{\Theta}, \quad (3.12)$$

both side of it multiply by  $(\ln \Theta)_{xx}$  and integrate in  $\mathbb{R}_+ \times (0, t)$  we can get

$$\begin{aligned} & \|(\ln \Theta)_x\|^2 + a \int_0^t \|(\ln \Theta)_{xx}\|^2 dt \\ & \leq \|(\ln \Theta_0)_x\|^2 + \int_0^t (\ln \Theta)_t (\ln \Theta)_x|_0^\infty dt. \end{aligned} \quad (3.13)$$

Then from (2.6) and (3.13) we can get

$$\|(\ln \Theta)_x\|^2 + a \int_0^t \|(\ln \Theta)_{xx}\|^2 dt \leq C\delta_0. \quad (3.14)$$

Then, if  $\int_0^t \int_{\mathbb{R}_+} ((2.5)_1 - (3.1)_1) \times (\Theta - \theta_2) dx dt$  combine with Cauchy-Schwarz inequality we can get

$$\|\Theta - \theta_2\|^2 + \int_0^t \|(\ln \Theta)_x\|^2 dt \leq C \int_0^t \|\theta_{2x}\|^2 dt. \quad (3.15)$$

Use (3.2) to (3.15) we can get

$$\|\Theta - \theta_2\|^2 + \int_0^t \|(\ln \Theta)_x\|^2 dt \leq C(1+t)^{1/3}. \quad (3.16)$$



That is (3.6).

Next, from

$$\int_0^t \int_{\mathbb{R}_+} (2.5)_1 \times \Theta^{-1} (\ln \Theta)_{xx} (1+t) dx dt,$$

we can get

$$\begin{aligned} & \int_0^t (1+t) ((\ln \Theta)_t (\ln \Theta)_x) (0, t) dt \\ &= a \int_0^t \int_0^\infty \frac{(\ln \Theta)_{xx}^2}{\Theta} (1+t) dx dt + \int_0^t \int_0^\infty ((\ln \Theta)_x)_t (1+t) dx dt. \end{aligned} \quad (3.17)$$

Because

$$\int_0^t (1+t) (\ln \Theta)_t (\ln \Theta)_x (0, t) dt = 0, \quad (3.18)$$

we can get

$$\begin{aligned} & (1+t) \|(\ln \Theta)_x\|^2 + \int_0^t \int_0^\infty (1+t) (\ln \Theta)_{xx}^2 dx dt \\ & \leq C \|\Theta_{0x}\|^2 + \int_0^t \int_0^\infty (\ln \Theta)_x^2 dx dt. \end{aligned} \quad (3.19)$$

Combine with (3.16) we can get

$$\begin{aligned} & (1+t) \|(\ln \Theta)_x\|^2 + \int_0^t \int_0^\infty (1+t) (\ln \Theta)_{xx}^2 dx dt \\ & \leq C(1+t)^{1/3}. \end{aligned} \quad (3.20)$$

That means  $\|(\ln \Theta)_x\|^2 \leq C(1+t)^{-2/3}$ , which is (3.7).

Again from (2.5)<sub>1</sub> we can get

$$(\ln \Theta)_{xt} = a \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x. \quad (3.21)$$

Both side of (3.21) multiply  $\partial_x^3 \ln \Theta$  and get

$$((\ln \Theta)_{xt} \partial_x^2 (\ln \Theta))_x - 1/2 (\partial_x^2 \ln \Theta)_t = a \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x \partial_x^3 (\ln \Theta). \quad (3.22)$$

Because

$$\begin{aligned} & ((\ln \Theta)_{xt} \partial_x^2 (\ln \Theta))_x (1+t)^2 \\ &= ((\ln \Theta)_{xt} (\ln \Theta)_{xx})_x (1+t)^2 \\ &= a^{-1} ((\ln \Theta)_{xt} \Theta_t)_x (1+t)^2, \end{aligned}$$

both side of (3.22) multiply  $(1+t)^2$ , then integrate in  $\mathbb{R}_+ \times (0, t)$  and combine with  $\Theta_t(0, t) = 0$ ,  $\Theta_t(\infty, t) = 0$ ,  $\Theta_x(\infty, t) = 0$  and Cauchy-Schwarz inequality to get for some small  $\epsilon > 0$  we have

$$0 \geq a \int_0^t \int_0^\infty \frac{(\ln \Theta)_{xxx}^2}{\Theta} (1+t)^2 dx dt$$

$$\begin{aligned}
& -\epsilon \int_0^t \int_0^\infty (1+t)^2 (\ln \Theta)_{xxx}^2 dx dt - C\epsilon^{-1}a \int_0^t \int_0^\infty (1+t)^2 (\ln \Theta)_{xx}^2 (\ln \Theta)_x^2 dx dt \\
& + 1/2 \|(\ln \Theta)_{xx}\|^2 (1+t)^2 - 1/2 \|(\ln \Theta_0)_{xx}\|^2 - \int_0^t \|(\ln \Theta)_{xx}\|^2 (1+t) dx \\
& \geq Ca \int_0^t \int_0^\infty \frac{(\ln \Theta)_{xxx}^2}{\Theta} (1+t)^2 dx dt \\
& - C\epsilon^{-1}a \int_0^t \int_0^\infty (1+t)^2 \|(\ln \Theta)_{xx}\| \|(\ln \Theta)_{xxx}\| (\ln \Theta)_x^2 dx dt \\
& + 1/2 \|(\ln \Theta)_{xx}\|^2 (1+t)^2 - 1/2 \|(\ln \Theta_0)_{xx}\|^2 - \int_0^t \|(\ln \Theta)_{xx}\|^2 (1+t) dx. \tag{3.23}
\end{aligned}$$

Take (3.20) into (3.23) we can get

$$\begin{aligned}
& \|(\ln \Theta)_{xx}\|^2 (1+t)^2 + \int_0^t \int_0^\infty (1+t)^2 (\ln \Theta)_{xxx}^2 dx dt \\
& \leq C(1+t)^{1/3}, \tag{3.24}
\end{aligned}$$

which also means

$$\|(\ln \Theta)_{xx}\|^2 \leq C(1+t)^{-5/3}, \tag{3.25}$$

so we finish (3.8).

If both side of (3.22) multiply by  $(1+t)$ , similar as the proof of (3.24), when combine with (3.14) we can get

$$\|(\ln \Theta)_{xx}\|^2 (1+t) + \int_0^t \int_0^\infty (1+t) (\partial_x^3 \ln \Theta)^2 dx dt \leq C\delta_0^2, \tag{3.26}$$

which means (3.9).

From (3.21) we can get

$$\partial_t (\ln \Theta)_{xx} = a \partial_x^2 \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right). \tag{3.27}$$

Because

$$((\ln \Theta)_{xxt} (\ln \Theta)_{xxx})_x = (a^{-1} \Theta_{tt} (\ln \Theta)_{xxx})_x,$$

when both side of (3.27) multiply  $(\partial_x^4 \ln \Theta)(1+\tau)^3$ , then integrate in  $\mathbb{R}_+ \times (0, t)$ , we can get

$$\begin{aligned}
& \int_0^t \int_0^\infty (a^{-1} \Theta_{tt} (\ln \Theta)_{xxx})_x (1+\tau)^3 dx d\tau \\
& = \int_0^t \int_0^\infty a \partial_x^2 \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right) \partial_x^4 \ln \Theta (1+\tau)^3 dx d\tau \\
& \quad + \int_0^t \int_0^\infty \frac{1}{2} ((\partial_x^3 \ln \Theta)^2)_t (1+\tau)^3 dx d\tau. \tag{3.28}
\end{aligned}$$

So from (3.20) and (3.24), we can get that for a small  $\epsilon > 0$ , (3.28) can be change to

$$\|\partial_x^3 \ln \Theta\|^2 (1+t)^3 + C \int_0^t (1+\tau)^3 \|\partial_x^4 \ln \Theta\|^2 d\tau$$

$$\begin{aligned}
&\leq C + C \int_0^t \int_0^\infty (\partial_x^3 \ln \Theta)^2 (\ln \Theta)_x^2 (1 + \tau)^3 dx d\tau + C \int_0^t \int_0^\infty (\ln \Theta)_{xx}^4 (1 + \tau)^3 dx d\tau \\
&\quad + C \int_0^t \int_0^\infty (\partial_x^2 \ln \Theta)^2 (\ln \Theta)_x^4 (1 + \tau)^3 dx d\tau + C \int_0^t \int_0^\infty (\partial_x^3 \ln \Theta)^2 (1 + \tau)^2 dx d\tau \\
&\leq C \int_0^t \|(\ln \Theta)_x\|^2 \|\partial_x^3 \ln \Theta\| \|\partial_x^4 \ln \Theta\| (1 + \tau)^3 d\tau + C \int_0^t \|(\ln \Theta)_{xx}\|^3 \|\partial_x^3 \ln \Theta\| (1 + \tau)^3 d\tau \\
&\quad + \int_0^t \|(\ln \Theta)_{xx}\|^4 \|(\ln \Theta)_x\|^2 (1 + \tau)^3 d\tau + C(1 + t)^{1/3} \\
&\leq \epsilon \int_0^t \|\partial_x^4 \ln \Theta\|^2 (1 + \tau)^3 d\tau + C\epsilon^{-1} \int_0^t \|\partial_x^3 \ln \Theta\|^2 (1 + \tau)^2 d\tau \\
&\quad + C\epsilon^{-1} \int_0^t \|\partial_x^2 \ln \Theta\|^2 (1 + \tau) d\tau + C(1 + t)^{1/3}.
\end{aligned}$$

Again using (3.20) and (3.24) we can get

$$\|\partial_x^3 \ln \Theta\|^2 (1 + t)^3 + \int_0^t (1 + \tau)^3 \|\partial_x^4 \ln \Theta\|^2 d\tau \leq C(1 + t)^{1/3}. \quad (3.29)$$

This means (3.10) finished.

Similar as above, when both side of (3.27) multiply  $(\partial_x^4 \ln \Theta)(1 + \tau)^2$ , then integrate in  $\mathbb{R}_+ \times (0, t)$  we can get

$$\|\partial_x^3 \ln \Theta\|^2 (1 + t)^2 + \int_0^t (1 + \tau)^2 \|\partial_x^4 \ln \Theta\|^2 d\tau \leq C. \quad (3.30)$$

From (2.4) we can get

$$(\Theta - \Theta_0)_t (\Theta - \Theta_0) = a((\ln \Theta)_x (\Theta - \Theta_0))_x - a(\ln \Theta)_x (\Theta - \Theta_0)_x.$$

When integrate both sides of it integrate in  $\mathbb{R}_+ \times [0, t]$ , we can get

$$\|\Theta - \Theta_0\|^2 \leq C \|\Theta_{0x}\|_{L^1} \int_0^t \|\Theta_x\|_{L^\infty} d\tau \leq C \|\Theta_{0x}\|_{L^1} \int_0^t \|\Theta_x\|^{1/2} \|\Theta_{xx}\|^{1/2} d\tau. \quad (3.31)$$

From Lemma 2.1, (3.31), (3.7) and (3.8) we can obtain

$$\|\Theta - \theta_+\|^2 \leq C(1 + t)^{5/12} + \|\theta_+ - \Theta_0\|^2 \leq C(1 + t)^{5/12}. \quad (3.32)$$

So from (3.5) and (3.7),

$$\|\Theta - \theta_+\|_{L^\infty}^2 \leq C \|\Theta - \theta_+\| \|\Theta_x\|^{3/4} \|\Theta_x\|^{1/4} \leq C \delta_0^{1/4} (1 + t)^{-1/24}. \quad (3.33)$$

So we finish this lemma.  $\square$

We can obtain from  $|(V - v_+, U, \Theta - \theta_+)|^2(x, t) \leq C|(V - v_+, U, \Theta - \theta_+)| \|(V_x, U_x, \Theta_x)\|$ , (3.4) and Lemma 3.2 that for  $x \in \mathbb{R}_+$ ,

$$\lim_{t \rightarrow \infty} |(V, U, \Theta)|(x, t) = (v_+, 0, \theta_+). \quad (3.34)$$

If

$$\|(v - V, u - U, \theta - \Theta)\|_{L^\infty(\mathbb{R}_+)} \rightarrow 0, \quad t \rightarrow \infty,$$

we can get that the asymptotic stability results to  $(v, u, \theta)$  is  $(v_+, 0, \theta_+)$ . This stability result can be obtained at the end of the paper.

We shall prove Theorem 2.1 by combining the local existence and the global-in-time a priori estimates. Since the local existence of the solution is well known (see, for example, [21]), we omit it here for brevity. to prove the global existence part of Theorem 2.1, it is sufficient to establish the following a priori estimates.

**Proposition 3.1** (A priori estimate) *Let  $(\varphi, \psi, \zeta) \in X([0, t])$  be a solution of problem (2.16) for some  $t > 0$ . Set  $C$  is a positive constant only depends on  $C_v, R, \mu, \theta_\pm, v_\pm$  and  $\|(\varphi_0, \psi_0, \zeta_0)\|_1$ ,  $C_0 > 2(C\|(\varphi_0, \psi_0, \zeta_0)\|_1^2 + C + 1)^{1/2}$ . If  $\|(\varphi_0, \psi_0, \zeta_0)\|$  is a suitably small constant,*

$$\bar{N}_1(t) = \max\{m_\rho^{-1}, M_\rho, m_\theta^{-1}, M_\theta, \|(\varphi, \psi, \zeta)\|_1\} \leq C_0,$$

with  $0 < m_\rho = v^{-1}(x, t) \leq \rho(x, t) \leq M_\rho$ ,  $0 < m_\theta \leq \theta(x, \tau) \leq M_\theta$ , then  $(\varphi, \psi, \zeta)$  satisfies the a priori estimate

$$\|(\varphi, \psi, \zeta)\|_1^2 + \int_0^t \{\|\varphi_x\|^2 + \|(\psi_x, \zeta_x)\|_1^2\} d\tau \leq C\|(\varphi_0, \psi_0, \zeta_0)\|_1^2 + C < C_0^2/4, \quad (3.35)$$

and  $\bar{N}_1(t) \leq C_0/2$ .

## 4 Proof of Theorem 2.1

Under the preparations in last section, the main task here is to finish Proposition 3.1 by the following lemmas.

**Lemma 4.1** *If  $C(\delta_0) > 0$  is a small constant about  $\delta_0$*

$$\int_0^t \int_{\mathbb{R}_+} \Theta_x^2(\varphi^2 + \zeta^2) dx d\tau \leq C(\delta_0) \int_0^t \|(\varphi_x, \zeta_x)\|^2 d\tau.$$

*Proof.*

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+} \Theta_x^2(\zeta^2 + \varphi^2) dx d\tau \\ & \leq \int_0^t \int_{\mathbb{R}_+} \Theta_x^2(\|\zeta\|\|\zeta_x\| + \|\varphi\|\|\varphi_x\|) dx d\tau \\ & \leq C \int_0^t (\|\zeta_x\| + \|\varphi_x\|)^2 \|\Theta_x\|^{1/4} d\tau + C \int_0^t \|\Theta_x\|^{15/4} d\tau. \end{aligned}$$

From (3.7) and (3.5) we can get

$$\int_0^t \int_{\mathbb{R}_+} \Theta_x^2(\zeta^2 + \varphi^2) dx d\tau \leq C(\delta_0) \int_0^t (\|\zeta_x\| + \|\varphi_x\|)^2 d\tau + C(\delta_0).$$

That we finish this lemma.  $\square$

**Lemma 4.2** *If  $C(\delta_0) > 0$  is small constant about  $\delta_0$ , we can get*

$$\begin{aligned} & \int_{\mathbb{R}_+} (\varphi^2 + \psi^2 + \zeta^2) dx + \int_0^t \|(\psi_x, \zeta_x)\|^2 d\tau \\ & \leq C(\delta_0) + C(\delta_0) \int_0^t (\|\varphi_x\|^2 + \|\psi_{xx}\|^2) d\tau + C\|(\varphi_0, \psi_0, \zeta_0)\|^2. \end{aligned} \quad (4.1)$$

*Proof.* Set

$$\begin{aligned} \Phi(z) &= z - \ln z - 1, \\ \Psi(z) &= z^{-1} + \ln z - 1, \end{aligned}$$

where  $\Phi'(1) = \Phi(1) = 0$  is a strictly convex function around  $z = 1$ . Similar to the proof in [21], we deduce from (2.16) that

$$\begin{aligned} & \left( \frac{\psi^2}{2} + R\Theta\Phi\left(\frac{v}{V}\right) + C_v\Theta\Phi\left(\frac{\theta}{\Theta}\right) \right)_t \\ & + \mu \frac{\Theta\psi_x^2}{v\theta} + \kappa \frac{\Theta\zeta_x^2}{v\theta^2} + H_x + Q = \mu \left( \frac{\psi\psi_x}{v} \right)_x - F\psi - \frac{\zeta G}{\theta}, \end{aligned} \quad (4.2)$$

where

$$H = R \frac{\zeta\psi}{v} - R \frac{\Theta\varphi\psi}{vV} + \mu \frac{U_x\varphi\psi}{vV} - \kappa \frac{\zeta\zeta_x}{v\theta} + \kappa \frac{\Theta_x\varphi\zeta}{v\theta V},$$

and

$$\begin{aligned} Q &= p_+\Phi\left(\frac{V}{v}\right)U_x + \frac{p_+}{\gamma-1}\Phi\left(\frac{\Theta}{\theta}\right)U_x - \frac{\zeta}{\theta}(p_+ - p)U_x - \mu \frac{U_x\varphi\psi_x}{vV} \\ & - \kappa \frac{\Theta_x}{v\theta^2}\zeta\zeta_x - \kappa \frac{\Theta\Theta_x}{v\theta^2V}\varphi\zeta_x - 2\mu \frac{U_x}{v\theta}\zeta\psi_x + \kappa \frac{\Theta_x^2}{v\theta^2V}\varphi\zeta + \mu \frac{U_x^2}{v\theta V}\varphi\zeta \\ & =: \sum_{i=1}^9 Q_i. \end{aligned}$$

Note that  $p = R\theta/v$ ,  $p_+ = R\Theta/V$  and (2.4), use integrate by part and Cauchy-Schwarz inequality can get

$$\begin{aligned} Q_1 + Q_2 &= Ra \left( \Phi\left(\frac{V}{v}\right)(\ln \Theta)_x \right)_x + \frac{Ra}{\gamma-1} \left( \Phi\left(\frac{\Theta}{\theta}\right)(\ln \Theta)_x \right)_x \\ & - aR(\ln \Theta)_x \left( \frac{V\varphi_x\varphi - V_x\varphi^2}{Vv^2} \right) \\ & - a \frac{p_+}{\gamma-1} (\ln \Theta)_x \left( \frac{\Theta\zeta_x\zeta - \Theta_x\zeta^2}{\Theta\theta^2} \right) \\ & \geq \left( p_+\Phi\left(\frac{V}{v}\right)U + \frac{p_+}{\gamma-1}\Phi\left(\frac{\Theta}{\theta}\right)U \right)_x \\ & - \epsilon(\zeta_x^2 + \varphi_x^2) - C\epsilon^{-1}\Theta_x^2(\zeta^2 + \varphi^2). \end{aligned} \quad (4.3)$$

Similarly, using  $p - p_+ = \frac{R\zeta - p_+\varphi}{v}$ , we can get

$$Q_3 \geq \frac{R\zeta - p_+\varphi}{v} \left( \frac{\zeta}{\theta} U_x \right) \geq \left( \frac{R\zeta^2 U}{v\theta} - \frac{p_+\zeta\varphi U}{\theta v} \right)_x - C(\delta_0)(\zeta_x^2 + \varphi_x^2) - C^{-1/2}(\delta_0)\Theta_x^2(\zeta^2 + \varphi^2). \quad (4.4)$$

And

$$\begin{aligned}
(Q_4 + Q_7) + (Q_5 + Q_6 + Q_8) + Q_9 &\geq -CC^{-1/2}(\delta_0)(\ln \Theta)_{xx}^2 - C^{1/2}(\delta_0)\psi_x^2 \\
&\quad - C^{1/2}(\delta_0)\zeta_x^2 - CC^{-1/2}(\delta_0)\Theta_x^2(\zeta^2 + \varphi^2) \\
&\quad - CC^{-1/2}(\delta_0)|(\ln \Theta)_{xx}|^2(\zeta^2 + \varphi^2). \tag{4.5}
\end{aligned}$$

At the end we use the definition of  $F$  and  $G$  in (2.14) then combine with the general inequality skills as above to get

$$\begin{aligned}
-F\psi - G\frac{\zeta}{\theta} &= -\frac{\kappa a(\gamma - 1) - \mu p_+ \gamma}{R\gamma} \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x \psi \\
&\quad + \frac{\mu p_+}{R\Theta} \left( \frac{\kappa(\gamma - 1)}{R\gamma} (\ln \Theta)_{xx} \right)^2 \frac{\zeta}{\theta} \\
&\leq -\frac{\kappa a(\gamma - 1) - \mu p_+ \gamma}{R\gamma} \left( \frac{(\ln \Theta)_{xx}}{\Theta} \psi \right)_x + \frac{\kappa a(\gamma - 1) - \mu p_+ \gamma}{R\gamma} \frac{(\ln \Theta)_{xx}}{\Theta} \psi_x \\
&\quad + \frac{\mu p_+}{R\Theta} \left( \frac{\kappa(\gamma - 1)}{R\gamma} (\ln \Theta)_{xx} \right)^2 \frac{\zeta}{\theta} \\
&\leq -\frac{\kappa a(\gamma - 1) - \mu p_+ \gamma}{R\gamma} \left( \frac{(\ln \Theta)_{xx}}{\Theta} \psi \right)_x + C^{1/2}(\delta_0)\psi_x^2 + CC^{-1/2}(\delta_0)(\ln \Theta)_{xx}^2 \tag{4.6}
\end{aligned}$$

Integrating (4.3) to (4.6) over  $\mathbb{R} \times (0, t)$ , using Lemma 3.2 and the boundary condition about  $(\varphi, \psi, \zeta)$  of (2.16) to estimate the terms  $\mu \left( \frac{\psi \psi_x}{v} \right)_x$ ,  $\left( \frac{(\ln \Theta)_{xx} \psi}{\Theta} \right)_x$  and  $H_x$ , in the end combine with Cauchy-Schwarz inequality we know that for a small  $C(\delta_0) > 0$  which is about  $\delta_0$ , we have

$$\begin{aligned}
&\int_{\mathbb{R}_+} \left( R\Theta \Phi \left( \frac{v}{V} \right) + \frac{1}{2} \psi^2 + C_v \Theta \Phi \left( \frac{\theta}{\Theta} \right) \right) dx + \int_0^t \left\| \left( \psi_x / (\sqrt{v\theta}), \zeta_x / (\theta \sqrt{v}) \right) \right\|^2 d\tau \\
&\leq C^{-1/2}(\delta_0) \left\{ \int_0^t \int_0^{+\infty} \Theta_x^2(\varphi^2 + \zeta^2) dx d\tau + \|\Theta_{0x}\|^2 \right\} + C \left\{ C^{1/2}(\delta_0) \int_0^t \|\varphi_x\|^2 d\tau + \|(\varphi_0, \psi_0, \zeta_0)\|^2 \right\} \\
&\quad + C^{-1/2}(\delta_0) \int_0^t \psi^2(0, \tau) d\tau + C^{1/2}(\delta_0) \int_0^t \psi_x^2(0, \tau) d\tau + C^{-1/2}(\delta_0) \int_0^t (\ln \Theta)_{xx}^2(0, \tau) d\tau + C(\delta_0). \tag{4.7}
\end{aligned}$$

Using the definition about  $\psi(0, t)$  in (2.16), then combine with (2.13)<sub>5</sub>, Cauchy-Schwarz inequality and Lemma 3.2 we can get

$$\begin{aligned}
&\int_0^t \psi^2(0, \tau) d\tau + \int_0^t (\ln \Theta)_{xx}^2(0, \tau) d\tau \\
&\leq C \int_0^t (\|(\ln \Theta)_x\|^{1/2} \|(\ln \Theta)_x\|^{7/2} + \|(\ln \Theta)_{xx}\|^{1/15} \|(\ln \Theta)_{xx}\|^{19/15}) d\tau \\
&\quad + \int_0^t (\|(\ln \Theta)_{xx}\|^2 + \|\partial_x^3(\ln \Theta)\|^2) d\tau \\
&\leq C(\delta_0). \tag{4.8}
\end{aligned}$$

Because

$$\int_0^t \psi_x^2(0, \tau) d\tau \leq C \int_0^t (\|\psi_x\|^2 + \|\psi_{xx}\|^2) d\tau,$$

combine with (4.8) and Lemma 4.1, (4.7) can be change to (4.1).  $\square$

**Lemma 4.3** *If  $\epsilon$  is a positive constant,  $\delta > 0$  stands for a small constant about  $\|(\varphi_0, \psi_0, \zeta_0)\|$  and  $\delta_0$ , we can get*

$$\begin{aligned} & \|(\varphi, \psi, \zeta)\|^2 + \|(\psi_x, \zeta_x)\|^2 + \int_0^t \|(\psi_{xx}, \zeta_{xx})\|^2 d\tau \\ & \leq C\|(\psi_{0x}, \zeta_{0x})\|^2 + C\delta + C\epsilon^{-1} \int_0^t \|\varphi_x\|^2 d\tau. \end{aligned}$$

*Proof.* First to get the estimate of  $\|\psi_x(t)\|$ , multiply both side of (2.16)<sub>2</sub> by  $\psi_{xx}$  to get

$$\begin{aligned} & \left(\frac{\psi_x^2}{2}\right)_t + \mu \frac{\psi_{xx}^2}{v} = \mu \frac{\psi_x v_x}{v^2} \psi_{xx} + \mu \left(\frac{U_x \varphi}{vV}\right)_x \psi_{xx} \\ & - R \left(\frac{\Theta \varphi}{vV}\right)_x \psi_{xx} + R \left(\frac{\zeta}{v}\right)_x \psi_{xx} + F \psi_{xx} + (\psi_t \psi_x)_x := \sum_{i=1}^6 I_i. \end{aligned}$$

When we integrate it in  $\mathbb{R}_+ \times (0, t)$ , we get

$$\begin{aligned} & \|\psi_x(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \\ & \leq C\|\psi_{0x}\|^2 + C \sum_{i=1}^6 \left| \int_0^t \int_0^\infty I_i dx d\tau \right|. \end{aligned} \quad (4.9)$$

Now we deal with  $\iint |I_i| dx d\tau$  in the right side of (4.9). Using  $v = \varphi + V$ ,  $R\Theta/V = p_+$ , Lemma 3.2, we can get

$$\begin{aligned} & \int_0^t \int_0^{+\infty} |I_1| dx d\tau \leq C \int_0^t \int_0^{+\infty} |V_x| |\psi_x| |\psi_{xx}| dx d\tau + C \int_0^t \int_0^{+\infty} |\varphi_x| |\psi_x| |\psi_{xx}| dx d\tau \\ & \leq C \int_0^t \|V_x\| \|\psi_x\|_{L^\infty} \|\psi_{xx}\| d\tau + C \int_0^t \|\psi_x\|_{L^\infty} \|\varphi_x\| \|\psi_{xx}\| d\tau \\ & \leq C^{1/2}(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + C^{-1/2}(\delta_0) \int_0^t \|\psi_x\|^2 \|V_x\|^4 d\tau + C \int_0^t \|\psi_x\|^{1/2} \|\varphi_x\| \|\psi_{xx}\|^{3/2} d\tau \\ & \leq C(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0) \int_0^t \|\psi_x\|^2 d\tau + C^{-1/4}(\delta_0) \sup_t \|\varphi_x\|^4 \int_0^t \|\psi_x\|^2 d\tau. \end{aligned} \quad (4.10)$$

Because  $N_1(t) \leq C_0$  and  $\|(\varphi_0, \psi_0, \zeta_0)\|$  is small, we can get from Lemma 4.2 that

$$\int_0^t \int_{\mathbb{R}_+} |I_1| dx d\tau \leq C(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + C\delta \int_0^t \|\varphi_x\|^2 d\tau + C\delta \int_0^t \|\psi_{xx}\|^2 d\tau + \delta.$$

Next we use the definition of  $(V, U, \Theta)$  (see (2.4), (2.13) and (3.4)), Cauchy-Schwarz inequality and Lemma 3.2 to get

$$\begin{aligned} & \int_0^t \int_0^\infty |I_2| dx d\tau \\ & \leq C \int_0^t \int_0^\infty (|U_{xx}| |\varphi| + |U_x| |\varphi_x| + |U_x| |V_x| |\varphi| + |U_x| |\varphi| |\varphi_x|) |\psi_{xx}| dx d\tau \end{aligned}$$

$$\begin{aligned}
&\leq C^{1/2}(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + \frac{C}{C^{1/2}(\delta_0)} \int_0^t \|\varphi\|_{L^\infty}^2 \|U_{xx}\|^2 d\tau + \frac{C}{C^{1/2}(\delta_0)} \int_0^t \|U_x\|_{L^\infty}^2 \|\varphi_x\|^2 d\tau \\
&\quad + \frac{C}{C^{1/2}(\delta_0)} \int_0^t \|\varphi\|_{L^\infty}^2 \|V_x\|^2 \|U_x\|_{L^\infty}^2 d\tau + \frac{C}{C^{1/2}(\delta_0)} \int_0^t \|\varphi\|_{L^\infty}^2 \|U_x\|_{L^\infty}^2 \|\varphi_x\|^2 d\tau \\
&\leq C^{1/2}(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0) + C(\delta_0) \int_0^t \|\varphi_x\|^2 d\tau.
\end{aligned} \tag{4.11}$$

The same as (4.10) and (4.11), we use Lemma 4.1, the definition of  $F$  in (2.14), Lemma 3.2 and (3.4) we can get the estimates about  $I_3$  to  $I_5$  as following.

$$\begin{aligned}
&\int_0^t \int_0^\infty (|I_3| + |I_4| + |I_5|) dx d\tau \\
&\leq C \int_0^t \int_0^\infty (|\Theta_x \|\varphi| + |\Theta| \|\varphi_x| + |\Theta| \|V_x\| \|\varphi| + |\Theta| \|\varphi\| \|\varphi_x|) |\psi_{xx}| dx d\tau \\
&\quad + C \int_0^t \int_0^\infty (|\zeta_x| + |\zeta| \|V_x| + |\zeta| \|\varphi_x|) |\psi_{xx}| dx d\tau \\
&\quad + C^{1/2}(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + \frac{C}{C^{1/2}(\delta_0)} \int_0^t \|F\|^2 d\tau \\
&\leq C^{1/2}(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + CC^{-1/2}(\delta_0) \int_0^t \|\varphi_x\|^2 d\tau + \frac{C}{C^{1/2}(\delta_0)} \int_0^t \|\varphi_x\|^2 d\tau \\
&\quad + \frac{C}{C^{1/2}(\delta_0)} \int_0^t \int_0^\infty V_x^2 \varphi^2 dx d\tau + C^{1/2}(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + \frac{C}{C^{1/2}(\delta_0)} \int_0^t \int_0^\infty (\zeta_x^2 + V_x^2 \zeta^2) dx d\tau \\
&\quad + \frac{C}{C^{1/2}(\delta_0)} \sup_t \|(\varphi, \zeta)\| \|(\varphi_x, \zeta_x)\| \int_0^t \|\varphi_x\|^2 d\tau + C^{1/2}(\delta_0) \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0).
\end{aligned} \tag{4.12}$$

Because  $N_1(t) \leq C_0$  and  $\|(\varphi_0, \psi_0, \zeta_0)\|$  is small, we can get from Lemma 4.2 that

$$\frac{C}{C^{1/2}(\delta_0)} \sup_t \|(\varphi, \zeta)\| \|(\varphi_x, \zeta_x)\| \int_0^t \|\varphi_x\|^2 d\tau \leq C\delta \int_0^t \|\varphi_x\|^2 d\tau + C\delta \int_0^t \|\psi_{xx}\|^2 d\tau + \delta,$$

Therefore

$$\begin{aligned}
&\int_0^t \int_0^\infty (|I_3| + |I_4| + |I_5|) dx d\tau \\
&\leq \epsilon \int_0^t \|\psi_{xx}\|^2 d\tau + \frac{C}{\epsilon} \int_0^t \int_0^\infty (\zeta_x^2 + \varphi_x^2) dx d\tau + C\delta \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0) + \delta.
\end{aligned}$$

At last we use integration by parts to the term about  $I_6$  to get

$$\begin{aligned}
&\left| \int_0^t \int_0^\infty I_6 dx d\tau \right| = \left| \int_0^t (\psi_t \psi_x)(0, \tau) d\tau \right| \leq \frac{C}{C^{1/2}(\delta_0)} \int_0^t \psi_x^2(0, \tau) d\tau + C^{1/2}(\delta_0) \int_0^t \psi_\tau^2(0, \tau) d\tau \\
&\leq C^{-1/2}(\delta_0) \int_0^t \|\psi_x\|^2 d\tau + 1/16 \int_0^t \|\psi_{xx}\|^2 d\tau + C^{1/2}(\delta_0) \int_0^t \psi_\tau^2(0, \tau) d\tau.
\end{aligned} \tag{4.13}$$

Using the definition of  $U$  in (2.4),  $\psi = u - U$  and (3.26) to get

$$\psi_t(0, t) = -\frac{k(\gamma-1)}{\gamma R} (\ln \Theta)_{xt}(0, t)$$



$$= -a \frac{k(\gamma-1)}{\gamma R} \partial_x \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right) (0, t). \quad (4.14)$$

Combine with Lemma 3.2 and (3.30) we get

$$\begin{aligned} \int_0^t \|(\ln \Theta)_{x\tau}\|_{L^\infty}^2(0, \tau) d\tau &\leq C \int_0^t \|(\ln \Theta)_{xxx}\|^2 d\tau + C \int_0^t \|\partial_x^4(\ln \Theta)\|^2 d\tau \\ &\quad + C \int_0^t \|(\ln \Theta)_x\|_{L^\infty} \|(\ln \Theta)_{xx}\|_{L^\infty} d\tau \leq C. \end{aligned} \quad (4.15)$$

So combine with Lemma 4.2, (4.13), (4.14) and (4.15) we get

$$\begin{aligned} &\left| \int_0^t \int_0^\infty I_6 dx d\tau \right| \\ &\leq \frac{C}{C^{1/2}(\delta_0)} \int_0^t \|\psi_x\|^2 d\tau + 1/16 \int_0^t \|\psi_{xx}\|^2 d\tau + C(\delta_0) \\ &\leq \delta + C\delta \int_0^t \|\varphi_x\|^2 d\tau + 1/16 \int_0^t \|\psi_{xx}\|^2 d\tau. \end{aligned} \quad (4.16)$$

In all, there exist a small constant  $\delta$  which is about  $\|(\varphi_0, \psi_0, \zeta_0)\|$  and  $\delta_0$ , such that

$$\begin{aligned} &\int_0^t \int_0^{+\infty} \sum_{i=1}^6 |I_i| dx d\tau \\ &\leq \int_0^t 1/2 \|\psi_{xx}\|^2 d\tau + C\epsilon^{-1} \int_0^t \|\varphi_x\|^2 d\tau + C\delta. \end{aligned} \quad (4.17)$$

So (4.9) can be change to

$$\begin{aligned} &\|\psi_x(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \\ &\leq C\delta + C\epsilon^{-1} \int_0^t \|\varphi_x\|^2 d\tau + C\|\psi_{0x}\|^2. \end{aligned} \quad (4.18)$$

The estimate about  $\|\zeta_x\|$  is similar to  $\|\psi_x\|$ , use (2.16)<sub>3</sub> multiply  $\zeta_{xx}$  then integrate in  $Q_t = \mathbb{R}_+ \times (0, t)$  to get

$$\begin{aligned} &\|\zeta_x\|^2 + \int_0^t \|\zeta_{xx}\|^2 d\tau \\ &\leq C\|\zeta_{0x}\|^2 + C \int_0^t \int_0^\infty (\psi_x^2 + \zeta^2 \psi_x^2 + \zeta^2 U_x^2 + U_x^2 \varphi^2) dx d\tau \\ &\quad + C \int_0^t \int_0^{+\infty} |\zeta_x|(|\varphi_x| + |V_x|) |\zeta_{xx}| dx d\tau + C \int_0^t \int_0^\infty \left| \left( \frac{\Theta_x \varphi}{vV} \right)_x \right|^2 dx d\tau \\ &\quad + C \int_0^t \int_0^{+\infty} (U_x^4 + \psi_x^4) dx d\tau + C \int_0^t \|G\|^2 d\tau \\ &=: C\|\zeta_{0x}\|^2 + \sum_{i=1}^5 J_i. \end{aligned} \quad (4.19)$$

Use the same method as (4.10)–(4.13) and combine with Lemma 4.2

$$J_1 \leq C(1+N^2(t)) \int_0^t \|\psi_x\|^2 d\tau + CN^2(t) \int_0^t \|U_x\|^2 d\tau \leq C\delta \left( 1 + \int_0^t \|\psi_{xx}\|^2 d\tau + \int_0^t \|\varphi_x\|^2 d\tau \right) + C(\delta_0).$$

Again use the same method as (4.10)–(4.13) and combine with Lemma 4.2

$$\begin{aligned} J_2 &\leq C \int_0^t \|\zeta_x\|_{L^\infty} \|\varphi_x\| \|\zeta_{xx}\| d\tau + C \int_0^t \|V_x\| \|\zeta_x\|_{L^\infty} \|\zeta_{xx}\| d\tau \\ &\leq C \int_0^t \|\zeta_x\|^{1/2} \|\zeta_{xx}\|^{3/2} \|\varphi_x\| d\tau + 1/16 \int_0^t \|\zeta_{xx}\|^2 d\tau + C(\delta_0) \int_0^t \|\zeta_x\|^2 d\tau \\ &\leq 1/8 \int_0^t \|\zeta_{xx}\|^2 d\tau + C(\delta_0) \int_0^t \|\zeta_x\|^2 d\tau + C \sup_t \|\varphi_x\|^4 \int_0^t \|\zeta_x\|^2 d\tau \\ &\leq C\delta \left( 1 + \int_0^t \|\psi_{xx}\|^2 d\tau + \int_0^t \|\varphi_x\|^2 d\tau \right) + C(\delta_0) + 1/8 \int_0^t \|\zeta_{xx}\|^2 d\tau. \end{aligned}$$

Because

$$\begin{aligned} &\left| \left( \frac{\Theta_x \varphi}{vV} \right)_x \right|^2 \\ &= \left| \frac{\Theta_{xx} \varphi}{vV} + \frac{\Theta_x \varphi_x}{vV} + \frac{\Theta_x \varphi}{vV} \left( -\frac{V_x + \varphi_x}{v^2} - \frac{V_x}{V^2} \right) \right|^2 \\ &\leq C\Theta_{xx}^2 \varphi^2 + C\Theta_x^2 \varphi_x^2 + C\Theta_x^2 V_x^2 \varphi^2 + C\Theta_x^2 \varphi^2 \varphi_x^2, \end{aligned}$$

combine with  $R\Theta/V = p_+$ , use the same method as (4.10)–(4.13) to get

$$\begin{aligned} J_3 &\leq C \int_0^t \|\varphi\|_{L^\infty}^2 \|\Theta_{xx}\|^2 d\tau + C \int_0^t \|\Theta_x\|_{L^\infty}^2 \|\varphi_x\|^2 d\tau \\ &\quad + C \int_0^t \int_0^{+\infty} \Theta_x^2 V_x^2 \varphi^2 dx d\tau \\ &\leq C(\delta_0) \int_0^t \|\varphi_x\|^2 d\tau + C(\delta_0). \end{aligned}$$

Use the definition  $U$  and similar as (4.10) (4.11) that we combine with Lemma 3.2 to get

$$\begin{aligned} J_4 &\leq C(\delta_0) + C \int_0^t \|\psi_x\|_{L^\infty}^2 \|\psi_x\|^2 d\tau \\ &\leq C(\delta_0) + C \int_0^t \|\psi_x\|^3 \|\psi_{xx}\| d\tau \\ &\leq C(\delta_0) + C \int_0^t (\|\psi_x\|^2 \|\psi_x\|^4 + 1/16 \|\psi_{xx}\|^2) d\tau \\ &\leq C\delta N^4(t) \left( 1 + \int_0^t \|\psi_{xx}\|^2 d\tau + \int_0^t \|\varphi_x\|^2 d\tau \right) + C(\delta_0) + 1/16 \int_0^t \|\psi_{xx}\|^2 d\tau. \end{aligned}$$

Use the definition  $G$  in (3.1) combine with Lemma 3.2

$$J_5 = C \int_0^t \|G\|^2 d\tau \leq C(\delta_0).$$

Use the results from  $J_1$  to  $J_5$ , the inequality (4.19) can be change to

$$\begin{aligned} & \|\zeta_x\|^2 + \int_0^t \|\zeta_{xx}\|^2 d\tau \\ & \leq C\|\zeta_{0x}\|^2 + C\delta \left(1 + \int_0^t \|\psi_{xx}\|^2 d\tau + \int_0^t \|\varphi_x\|^2 d\tau\right) + 1/16 \int_0^t \|\psi_{xx}\|^2 d\tau. \end{aligned} \quad (4.20)$$

In fact when combine with Lemma 4.2–4.1, (4.18) and (4.20), it is easy to get

$$\begin{aligned} & \|(\varphi, \psi, \zeta)\|^2 + \|(\psi_x, \zeta_x)\|^2 + \int_0^t \|(\psi_{xx}, \zeta_{xx})\|^2 d\tau \\ & \leq C\|(\psi_{0x}, \zeta_{0x})\|^2 + C\delta + C\epsilon^{-1} \int_0^t \|\varphi_x\|^2 d\tau. \end{aligned}$$

□

**Lemma 4.4** *For a small  $C(\delta) > 0$  stands for constant about  $\|(\varphi_0, \psi_0, \zeta_0)\|$  and  $\delta_0$ , and  $C(\delta_0) > 0$  is a small constant about  $\delta_0$ , we can get*

$$\|\varphi_x\|^2 + \int_0^t \|\varphi_x\|^2 d\tau \leq C\|(\varphi_{0x}, \psi_{0x}, \zeta_{0x})\|^2 + C\delta. \quad (4.21)$$

*Proof.* Set  $\bar{v} = \frac{v}{V}$ , take it into (2.16)<sub>1</sub>, (2.16)<sub>2</sub> ( $p = R\theta/v$ ) to get

$$\psi_t + p_x = \mu \left( \frac{\bar{v}_x}{\bar{v}} \right)_t - F.$$

Both sides of last equation multiply  $\bar{v}_x/\bar{v}$  to get

$$\begin{aligned} & \left( \frac{\mu}{2} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 - \psi \frac{\bar{v}_x}{\bar{v}} \right)_t + \frac{R\theta}{v} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 + \left( \psi \frac{\bar{v}_t}{\bar{v}} \right)_x \\ & = \frac{\psi_x^2}{v} + U_x \left( \frac{1}{v} - \frac{1}{V} \right) \psi_x + \frac{R\zeta_x}{v} \frac{\bar{v}_x}{\bar{v}} - \frac{R\theta}{v} \left( \frac{1}{\Theta} - \frac{1}{\theta} \right) \Theta_x \frac{\bar{v}_x}{\bar{v}} + F \frac{\bar{v}_x}{\bar{v}}. \end{aligned} \quad (4.22)$$

Because  $v|_{x=0} = V|_{x=0} = v_-$ , we can get

$$\left( \psi \frac{\bar{v}_t}{\bar{v}} \right) \Big|_{x=0} = 0.$$

On the other hand if we integrate (4.22) in  $R_+ \times (0, t)$ , combine with Cauchy-Schwartz inequality, (4.22) is changed to

$$\begin{aligned} & \int_{\mathbb{R}_+} \left( \frac{\mu}{2} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 - \psi \frac{\bar{v}_x}{\bar{v}} \right) dx + \int_0^t \int_{\mathbb{R}_+} \frac{R\theta}{v} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 dx d\tau \\ & \leq C^{-1/2}(\delta_0) \left( \int_0^t \|(\zeta_x, \psi_x)\|^2 d\tau + \int_0^t \int_0^{+\infty} \Theta_x^2 (\varphi^2 + \zeta^2) dx d\tau \right) \\ & + C \int_0^t \int_0^{+\infty} U_x^2 \varphi^2 dx d\tau + C \int_0^t \int_0^{+\infty} |F|^2 dx d\tau + 1/2 \int_0^t \left\| \frac{\sqrt{R\theta}}{\sqrt{v}} \frac{\bar{v}_x}{\bar{v}} \right\|^2 d\tau. \end{aligned}$$

Furthermore, because  $C_1(\varphi_x^2) - C_2V_x^2 \leq (\frac{\bar{v}_x}{\bar{v}})^2 \leq C_3\varphi_x^2 + C_4V_x^2$  ( $C_1, C_2, C_3, C_4$  stands for constants about  $v$ ), combine with Lemma 4.1, Lemma 4.2 and Lemma 4.3 we can get

$$\int_0^t \|\varphi_x\|^2 d\tau + \|\varphi_x\|^2 \leq C\|(\varphi_{0x}, \psi_{0x}, \zeta_{0x})\|^2 + C\delta. \quad (4.23)$$

So we finish this lemma.  $\square$

From Lemma 4.2 to Lemma 4.4 we know when  $\delta_0$  and  $\|(\varphi_0, \psi_0, \zeta_0)\|$  suitably small there exist a suitably small positive constant  $\delta$  such that

$$\|(\varphi, \psi, \zeta)\|^2 + \int_0^t \|(\psi_x, \zeta_x)\|^2 d\tau \leq C\delta,$$

Then we can get

$$|v - V|^2 \leq \|\varphi\| \|\varphi_x\| \leq C\delta,$$

which means  $C_5 \leq |v| \leq C_6$ . Use this result to Lemma 4.3, we can get there exist a positive constant  $C$  independent of  $v(x, t), u(x, t)$  and  $\theta(x, t)$  such that

$$\|(\varphi_x, \psi_x, \zeta_x)\|^2 + \int_0^t \|(\psi_{xx}, \zeta_{xx})\|^2 \leq C\|(\varphi, \psi, \zeta)\|_1^2.$$

Similar as the estimates for the upper and lower of  $v$ , when we combine  $\|\zeta_x\| \leq C$  with Lemma 4.2, we can obtain  $C_7 \leq |\theta| \leq C_8$ . Here  $C_5, C_6, C_7$  and  $C_8$  are constants independent of  $v(x, t), u(x, t)$  and  $\theta(x, t)$ . So we finish Proposition 2.2.

To finish Theorem 2.1 now we will proof  $\sup_{x \in \mathbb{R}_+} |(\varphi, \psi, \zeta)| \rightarrow 0$ , as  $t \rightarrow \infty$ .

Because  $\int_0^{+\infty} \partial_x(2.16)_1 \times 2\varphi_x dx$  equals to

$$0 = 2 \int_0^\infty \varphi_x \psi_{xx} dx - \frac{d}{dt} \|\varphi_x\|^2, \quad (4.24)$$

use Cauchy-Schwarz inequality we get

$$2 \int_0^\infty \varphi_x \psi_{xx} dx \leq C (\|\varphi_x\|^2 + \|\psi_{xx}\|^2),$$

again using Lemma 4.3–4.4 and (4.24), then we get

$$\begin{aligned} & \int_0^\infty \left| \frac{d}{dt} \|\varphi_x(t)\|^2 \right| dt \\ & \leq C \int_0^\infty (\|\varphi_x\|^2 + \|\psi_{xx}\|^2) dt \\ & \leq C\|(\varphi_0, \psi_0, \zeta_0)\|_1^2 + C\delta. \end{aligned} \quad (4.25)$$

Similar as above, from Lemma 4.2–4.4 and combine with Sobolev inequality we get

$$\int_0^\infty \left( \left| \frac{d}{dt} \|\psi_x(t)\|^2 \right| + \left| \frac{d}{dt} \|\zeta_x(t)\|^2 \right| \right) d\tau \leq C\|(\varphi_0, \psi_0, \zeta_0)\|_1^2 + C\delta. \quad (4.26)$$

It means

$$\|(\varphi, \psi, \zeta)(t)\|_{L^\infty}^2 \leq 2\|(\varphi, \psi, \zeta)(t)\| \|(\varphi_x, \psi_x, \zeta_x)(t)\| \rightarrow 0 \quad \text{when } t \rightarrow \infty.$$

Now when we combine with (3.34) we finish the theorem.

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